Math 10B with Professor Stankova
Worksheet, Discussion #2; Tuesday, 1/29/2019
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Complementary Counting and PIE

Concepts

1. **Complementary counting** is the problem solving technique (PST) of counting the opposite of what we want and subtracting that from the total number of cases. For instance, instead of counting the number of strings of 6 letters with at least one "E", we would subtract the number of strings without an "E" (the opposite) from total number of 6 letter strings.

   Principle of Inclusion/Exclusion (PIE) is another PST which corrects double counting. It says $|A \cup B| = |A| + |B| - |A \cap B|$ for two sets, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. In general, we add the size of all the sets, then subtract all possible pairs of sets, then add back all triplets of sets, subtract all four-sets of sets, etc.

Examples

2. Out of 200 students, there are 100 taking Calculus, 70 taking algebra, and 30 taking both. How many students are taking neither?

   Solution: There are $100 + 70 - 30 = 140$ students taking Calculus or algebra. Thus, there are $200 - 140 = 60$ taking neither.

3. How many 3 letter sequences do not contain the same letter twice in a row?

   Solution: There are a total of 26^3 3 letter sequences. Then we count the number of ways for the sequence to have at least two in a row. If these two in the row occur in the first two, this gives 26 for the selection of the repeated letter, then 26 for the other letter. Thus we get 26^2 and the same holds true if the two repeated letters are at the end. Then we add back the intersections which is the case when all the letters are the same and there are 26 such ones. So the final answer is

   $$26^3 - 26^2 - 26^2 + 26.$$

Problems

4. True **FALSE** We can only use the Principle of Inclusion/Exclusion if there are two or three cases or circles.

> **Solution:** We can do it with any number of circles/cases.

5. True **FALSE** Venn diagrams with two circles always look like interlocking rings.

> **Solution:** A circle inside another is a Venn diagram.

6. Last semester, out of all the students who took both intro chem and 10A, 75% of students passed the intro chem final and 85% passed the 10A final and 70% passed both. There were 50 students who failed both. How many total students took both intro chem and 10A?

> **Solution:** There are $75 + 85 - 70 = 90\%$ of students who passed at least one of the finals. Thus, there are $100 - 90 = 10\%$ that failed. So 50 students is 10% and hence there are 500 students.

7. How many numbers from 1 to 300 are even but not divisible by 3?

> **Solution:** There are 150 even numbers at 50 that are divisible by 6. Thus, there are $150 - 50 = 100$ that are even but not divisible by 3.

8. How many license plates with 3 letters followed by 3 digits have either the 3 letters forming a palindrome or the 3 digits forming a palindrome (or both)?

> **Solution:** There are $26^2 \cdot 10^3$ plates with a palindrome letter set and $26^3 \cdot 10^2$ with a palindrome number and $26^2 \cdot 10^2$ with both. Thus, there are $26^2 \cdot 10^3 + 26^3 \cdot 10^2 - 26^2 \cdot 10^2 = 26^2 \cdot 10^2(26 + 10 - 1) = 35 \cdot 26^2 \cdot 10^2$ different plates.

9. How many numbers less than or equal to 1000 are divisible by 7 or 11 but not both?

Solution: There are $\lfloor 1000/7 \rfloor = 142$ numbers divisible by 7. There are $\lfloor 1000/11 \rfloor = 90$ numbers divisible by 11. There are $\lfloor 1000/77 \rfloor = 12$ numbers divisible by both. Thus, $142 + 90 - 2 \cdot 12 = 208$ are divisible by 7 or 11 but not both.

10. How many four digit numbers do not have any repeating 1s?

Solution: There are a total of $9 \cdot 10^3$ four digit numbers. The bad cases are $11XX, X11X, XX11$. The cases have $100, 90, 90$ possibilities. The intersections are $111X, 1111, X111$ which give $10, 1, 9$ cases. Finally the intersection of all three is 1111 which has 1 case. Thus the total number is

$$9000 - 100 - 90 - 90 + 10 + 1 + 9 - 1.$$

11. How many ways are there to put 7 balls in 3 boxes if each box must have at least one ball?

Solution: There are 3^7 ways to put the 7 balls in 3 boxes. Let A, B, C be the cases where box $1, 2, 3$ are empty respectively. Then the number of ways for each is 2^7 and the intersections happen in a unique way. Thus, the total number of ways is

$$3^7 - 3 \cdot 2^7 + 3.$$

12. (Challenge) How many ways can we form 2 separate teams out of 5 people if not everyone needs to be on a team but the teams have to have at least one person (e.g. Team A could have person $1, 3$ and team B could have person 5 and person $2, 4$ do not have a team).

Solution: Each person number 1 through 5 can either be in team A or B or neither. Thus, there are 3^5 different ways to assign them. But, there 2^5 ways for team A to be empty, 2^5 for B to be empty. And finally 1 way for both to be empty. Thus, there are a total of $3^5 - 2 \cdot 2^5 + 1$ different ways.

Pigeonhole Principle

13. I have 7 pairs of socks in my drawer, one of each color of the rainbow. How many socks do I have to draw out in order to guarantee that I have grabbed at least one pair?

What if there are likewise colored pairs of gloves in there and I cannot tell the difference between gloves and socks and I want a matching set?

Solution: After grabbing 7 socks, worst case scenario, I have grabbed a sock of each color. Thus, after grabbing one more sock, it has to match up with one of the previous socks so after grabbing 8 socks I am guaranteed to have a pair.

For the second part, after grabbing 21 objects, it is possible that I have grabbed 3 items for each color and hence have gotten no sets yet. But the 22nd thing I grab must complete one of these 7 sets so after 22 items, I am guaranteed to have a matching set.